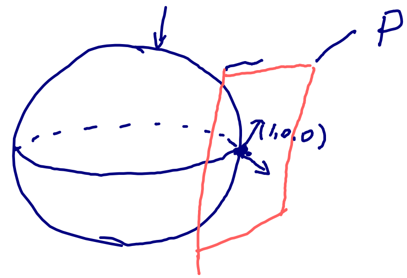
Math 2010 B. Tutorial 8.

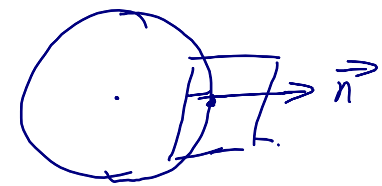
Outline:

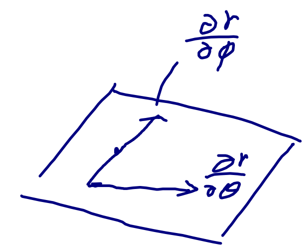
- Total differential
- chain Rule.

Q1: Find the tangent plane to the sphere

$S: \underline{x^2+y^2+z^2=1} \subseteq \mathbb{R}^3$ at $(1,0,0)$ in $\mathbb{R}^3$







"parametrization" & "Equation" for discribe plane.

1) parametrization method.

one point $(1,0,0)$

The map $\vec{r}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ $\rightarrow$ $\frac{\partial r}{\partial \phi}$, $\frac{\partial r}{\partial \theta}$

parametrizes the sphere S nere $(1,0,0)$ w/ $\vec{r}(\frac{\pi}{2},0) = (1,0,0)$

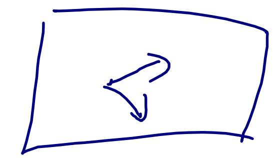
$$\frac{\partial \vec{r}}{\partial \phi}(\tfrac{\pi}{2},0) = (\cos\tfrac{\pi}{2}\cos 0, \cos\tfrac{\pi}{2}\sin 0, -\sin\tfrac{\pi}{2}) = (0,0,-1)$$

$$\frac{\partial \vec{r}}{\partial \theta}(\tfrac{\pi}{2},0) = (-\sin\tfrac{\pi}{2}\sin 0, \sin\tfrac{\pi}{2}\cos 0, 0) = (0,1,0)$$

The tangent plane.

one point $(1,0,0)$. & $\vec{v}_1 = (1,0,0)$ $\qquad$ $\vec{v}_2 = (0,1,0)$

$$\vec{v}_1 = \frac{\partial r}{\partial \phi}, \qquad \vec{v}_2 = \frac{\partial r}{\partial \theta}$$

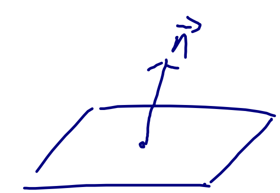


The required tangent plane is

$$\left\{ \vec{r}\left(\tfrac{\pi}{2},0\right) + s\,\frac{\partial r}{\partial \phi}\left(\tfrac{\pi}{2},0\right) + t\,\frac{\partial r}{\partial \theta}\left(\tfrac{\pi}{2},0\right) : s,t \in \mathbb{R} \right\}$$

$$\vec{r}\left(\tfrac{\pi}{2},0\right) = (1,0,0)$$

$$= \left\{ (1,\ t,\ -s) \in \mathbb{R}^3 ,\ s,t \in \mathbb{R} \right\}$$



Method 2: "Level Set" Method $\longleftrightarrow$ one point + one normal vector

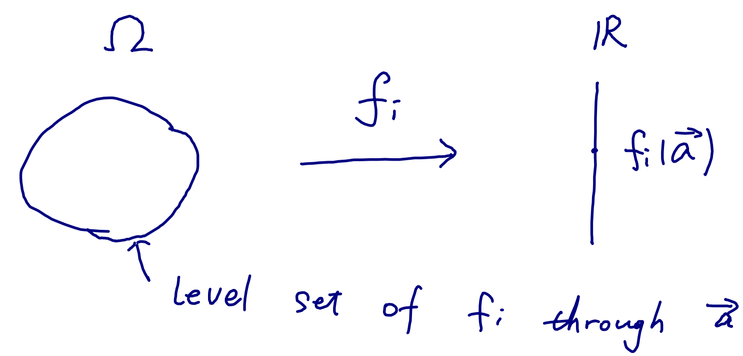
Review :

Let $\mathbb{R}^n \underset{\text{open}}{\supset} \Omega \xrightarrow{\vec{f} = (f_1, \cdots, f_m)} \mathbb{R}^m$, $\vec{a} \in \Omega$ s.t

$\frac{\partial f}{\partial x_j}(\vec{a})$ exists $\forall$ i.j

For $1 \le i \le n$, the gradient vector of f_i at $\vec{a}$ is

$$\vec{\nabla} f_i(\vec{a}) = \begin{pmatrix} \frac{\partial f_i}{\partial x_1}(\vec{a}) \\ \vdots \\ \frac{\partial f_i}{\partial x_n}(\vec{a}) \end{pmatrix} \in \mathbb{R}^n .$$



Level set of f_i through $\vec{a}$: $\left\{ \vec{x} \in \Omega : f_i(\vec{x}) = \underset{\sim}{f_i(\vec{a})} \right\}$

$\because$ $\vec{\nabla} f_i(\vec{a})$ is normal to the level set of f_i through $\vec{a}$.

The <u>total differential</u> of f_i at $\vec{a}$ is

$$df_i(\vec{a}) = \frac{\partial f_1}{\partial x_1}(\vec{a})\, dx_1 + \cdots + \frac{\partial f_n}{\partial x_n}(\vec{a})\, dx_n$$

As a linear map $\mathbb{R}^n \xrightarrow{df_i(\vec{a})} \mathbb{R}$

$$df_i(\vec{a})(\underset{\mathbb{R}^n}{\underset{\cap}{\vec{x}}}) = \vec{\nabla} f_i(\vec{a}) \cdot \vec{x} \qquad \forall\, x \in \mathbb{R}^n.$$

— The Jacbian matrix of f at $\vec{a}$ is

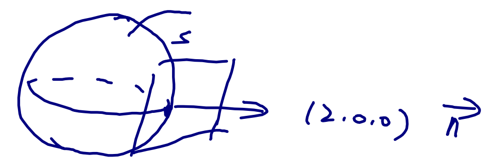
$$Df(\vec{a}) = \begin{pmatrix} \vec{\nabla} f_1 \\ \vdots \\ \vec{\nabla} f_m \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ & \vdots & & \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \frac{\partial f_m}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix}.$$

· return to method 2

② Level set Method

(ϕ, θ)

$r = 1$



Define $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$f(x,y,z) = x^2 + y^2 + z^2 \qquad (x,y,z) \in \mathbb{R}^3$$

In fact The sphere S is the level set of f through $(1,0,0)$

$\because \; f(1,0,0) = 1^2 + 0 + 0 = 1$

$$\{(x,y,z) \mid f(x,y,z) = f(1,0,0) = 1\} = S$$

$\nabla f(1,0,0)$ normal vector.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x \Rightarrow \frac{\partial f}{\partial x}(1,0,0) = 2 \\ \frac{\partial f}{\partial y} &= 2y \Rightarrow \frac{\partial f}{\partial x}(1,0,0) = 0 \\ \frac{\partial f}{\partial z} &= 2z \Rightarrow \frac{\partial f}{\partial z}(1,0,0) = 0 \end{aligned}\right\} \Rightarrow \nabla f(1,0,0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)(1,0,0)$$

$$= (2,0,0)$$

The required tangent plane is

normal vector $\vec{p}$

$$\{\vec{x} \in \mathbb{R}^3 . \boxed{\nabla f(1,0,0)} \cdot (\vec{x} - (1,0,0)) = 0\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : x = 1\}$$

define plane.

$$\{\vec{n} \cdot (\vec{x} - \vec{p}) = 0\}$$

Chain Rule :

$$\begin{array}{ccccc} \vec{a} & \longmapsto & f(\vec{a}) & \longmapsto & g(f(\vec{a})) = \underline{(g\circ f)(\vec{a})} \\ \mathbb{R}^k & \xrightarrow{f} & \mathbb{R}^n & \xrightarrow{g} & \mathbb{R}^m \\ \cup & & \cup & & \\ \Omega_1 & & \Omega_2 & & \end{array}$$

Thm :

Let $f : \Omega_1 \subset \mathbb{R}^k \longrightarrow \mathbb{R}^n$ be differentiable at $\vec{a} \in \Omega_1$

$g : \Omega_2 \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $f(\vec{a}) = \vec{b} \in \Omega_2$

Then $\vec{g}\circ\vec{f}$ is differentiable at $\vec{a}$ and

$$D(\vec{g}\circ\vec{f})(\vec{a}) = D\vec{g}(f(\vec{a}))\cdot D\vec{f}(\vec{a})$$

skip proof

e.g. change of coordinates.

Consider the map $\vec{g}(r,\theta) = (r\cos\theta, r\sin\theta)$.

Ex : Show that $\vec{g}$ is differentiable.

Let $\vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by

$$\vec{f}(x,y) = (e^x, y, x^2+y^2).$$

$\vec{a} = (\sqrt{2}, \frac{\pi}{4})$

show that

$D(f\circ g)(\vec{a}) = D\vec{f}(g(\vec{a}))\, D\vec{g}(\vec{a})$

$x = r\cos\theta$
$y = r\sin\theta$

$$f\circ g(r,\theta) = (e^{r\cos\theta}, r\sin\theta, r^2).$$

Remark:

$$\begin{array}{ccccccc} & & (x,y) & \longmapsto & (e^x, y, x^2+y^2) & & \\ \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^3 & \longrightarrow & (f\circ g)(r,\theta) \\ (r,\theta) & \longmapsto & (r\cos\theta, r\sin\theta) & & & & \| \\ & & & & & & (e^{r\cos\theta}, r\sin\theta, r^2) \end{array}$$

$$\vec{a} = (\sqrt{2}, \tfrac{\pi}{4}) \longmapsto (1,1) = g(\vec{a})$$

compute: $\underline{D\vec{g}(\vec{a})} = D\vec{g}(\sqrt{2}, \frac{\pi}{4})$

$\underline{D\vec{f}(g(\vec{a}))} = D\vec{f}\,(1,1)$

$D(f\circ g)(\vec{a}) = D(f\circ g)(\sqrt{2}, \frac{\pi}{4})$

$\Rightarrow$ show that $\underline{D(f\circ g)}(\vec{a}) = Df(g(\vec{a}))\, D\vec{g}(\vec{a})$